$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{h}{kA_c}\frac{dA_s}{dx}\right)(T - T_\infty) = 0$$

The general equation simplified to

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0 \qquad ....(1)$$

To simplify the form of this equation, we transform the dependent variable by defining an excess temperature θ as

$$\theta(x) = T(x) - T_\infty \qquad .... (2)$$

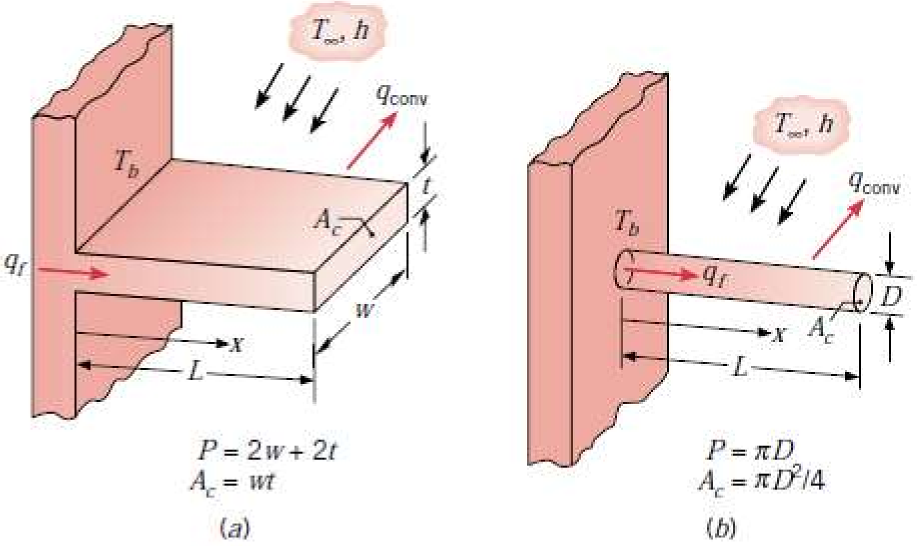


FIGURE 2.15 Straight fins of uniform cross section. (a) Rectangular fin. (b) Pin fin.

Therefore

$$\frac{d^2T}{dx^2} = \frac{d^2\theta}{dx^2}$$

Let, $\frac{hP}{kA_c} = m^2$

So equation (1) simplified to:

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad .... (3)$$

Equation (3) is a linear, homogeneous, second-order differential equation with constant coefficients. Its general solution is of the form

$$\theta(x) = C_1e^{-mx} + C_2e^{mx} \qquad .... (4)$$

To evaluate the constants C_1 and C_2 of Equation (4), it is necessary to specify appropriate boundary conditions.

Baundary condition: at $x = 0$ $T(x) = T_b$, subtitute in equation (2)

$\theta(0) = \theta_b = T_b - T_\infty$ .... (5)

Second baundary condition at $x = L$ there is four cases:

Case A: very long fin and the temperature of the tip of the fin approach the environmental temperature.

Boundary condition 1: At $x = 0, \theta(0) = \theta_b = T_b - T_\infty$

Substitute in equation (4) gives:

$\theta_b = C_1 + C_2$ .... (6)

Boundary condition 2: at $x = \infty$ $T(x) = T_\infty$, substitute in equation (2) gives:

$\theta = 0$ subtitute in equation (4) gives:

$0 = C_1 e^{-\infty} + C_2 e^{\infty}$

So, $C_2 = 0$, subtitute in equation (6) gives:

$C_1 = \theta_b$

Substitute C_1 and C_2 in equation (4) gives:

$\theta(x) = \theta_b e^{-mx}$

$\frac{\theta(x)}{\theta_b} = \frac{T(x)-T_\infty}{T_b-T_\infty} = e^{-mx}$ .... (7) (Temperature distribution)

To find the heat loss from the fin:

$q = -kA_c \frac{dT}{dx}$ .... (8)

At $x = 0$, $\frac{dT}{dx} = (T_b - T_\infty)(-me^0)$

Substitute in (8)

$q_f = kA_c m\theta_b = \sqrt{hPkA_c}\theta_b$ .... (9)

Case B: The assumption that the convective heat loss from the fin tip is negligible, in which case the tip may be treated as adiabatic and :

At $x = L$, $\frac{dT}{dx} = 0 = \frac{d\theta}{dx}$

$\theta(x) = C_1 e^{-mx} + C_2 e^{mx}$

$\frac{d\theta}{dx} = 0 = -mC_1 e^{-mL} + mC_2 e^{mL}$ .... (10)

$C_1 e^{-mL} = C_2 e^{mL}$

$C_1 = C_2 e^{2mL}$

At $x = 0$, $\theta_b = C_1 + C_2$

$\theta_b = C_1 + C_2 = C_2 e^{2mL} + C_2$

$C_2 = \frac{\theta_b}{1+e^{2mL}}$

$C_1 = \frac{\theta_b(e^{2mL})}{1+e^{2mL}}$

$C_1 = \frac{\theta_b}{1+e^{-2mL}}$

Substitute C_1 and C_2 in

$\theta(x) = \frac{\theta_b}{1+e^{-2mL}} e^{-mx} + \frac{\theta_b}{1+e^{2mL}} e^{mx}$

$\frac{\theta(x)}{\theta_b} = \frac{e^{-mx}}{1+e^{-2mL}} + \frac{e^{mx}}{1+e^{2mL}}$

$\frac{\theta(x)}{\theta_b} = \frac{e^{-mx} e^{2mL}}{1+e^{2mL}} + \frac{e^{mx}}{1+e^{2mL}}$

$\frac{\theta(x)}{\theta_b} = \frac{e^{-mx} e^{2mL} + e^{mx}}{1+e^{2mL}} * \frac{\frac{1}{e^{mL}}}{\frac{1}{e^{mL}}}$

$\frac{\theta(x)}{\theta_b} = \frac{e^{-mx}.e^{mL} + e^{mx}.e^{-mL}}{e^{-mL} + e^{mL}} = \frac{e^{m(L-x)} + e^{-m(L-x)}}{e^{mL} + e^{-mL}}$

$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh[m(L-x)]}{\cosh[mL]}$ .... (11)

Where:

$$\sinh x = \frac{e^{x}-e^{-x}}{2}$$

$$\cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$\tanh x = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

To find the heat loss from the fin:

$$q = -kA_c \frac{dT}{dx}$$

$$q_f = -kA_c \frac{d\theta}{dx}$$

$$\frac{d\theta}{dx} = \theta_b(-m)\frac{\sinh m(L-x)}{coshmL}$$

At $x = 0$,

$$\frac{d\theta}{dx} = -\theta_b\, m \tanh mL$$

$$q_f = k\, A_c\, m\, \theta_b \tanh mL$$

$$q_f = \sqrt{hPk\, A_c}\;\; \theta_b \tanh mL \qquad \text{.............} \quad (12)$$

Case C: finite length with heat convection from the fin tip.

considers convection heat transfer from the fin tip. Applying an energy balance to a control surface about this tip (Figure 2.16), we obtain

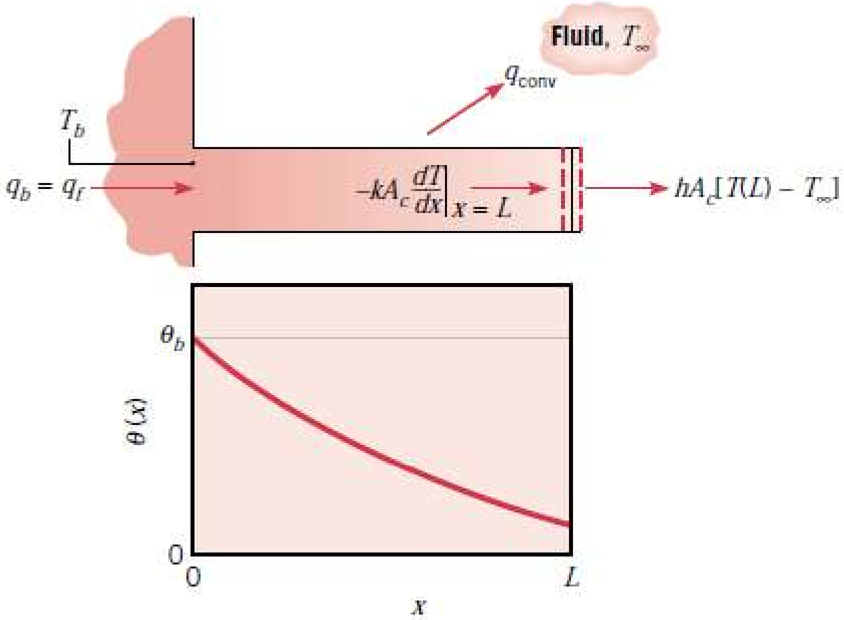


FIGURE 2.16 Conduction and convection in a fin of uniform cross section.

That is, the rate at which energy is transferred to the fluid by convection from the tip must equal the rate at which energy reaches the tip by conduction through the fin.

$\theta(x) = C_1 e^{-mx} + C_2 e^{mx}$

At x=0

$\theta_b = C_1 + C_2$ .... (13)

At $x = L$

$q_{cond} = q_{conv}$

$-kA_c \frac{d\theta}{dx} = hA_c(T(L) - T_\infty)$

$-kA_c(-mC_1 e^{-mL} + mC_2 e^{mL}) = hA_c(C_1 e^{-mL} + C_2 e^{mL})$

$C_1 e^{-mL} - C_2 e^{mL} = \frac{h}{km}(C_1 e^{-mL} + C_2 e^{mL})$ .... (14)

From equations (13) and (14) solve to find C_1 and C_2

$C_1 \left(e^{-mL} - \frac{h}{km} e^{-mL}\right) = C_2 \left(\frac{h}{km} e^{mL} + e^{mL}\right)$

$$C_1 = C_2 \frac{\left(\frac{h}{km}e^{mL}+e^{mL}\right)}{\left(e^{-mL}-\frac{h}{km}e^{-mL}\right)} \qquad \text{..........} \ (15)$$

Sub. Equation (15) in equation (13):

$$\theta_b = C_2 \frac{\left(\frac{h}{km}e^{mL}+e^{mL}\right)}{\left(e^{-mL}-\frac{h}{km}e^{-mL}\right)} + C_2$$

$$C_2 = \frac{\theta_b}{1+\frac{\left(\frac{h}{km}e^{mL}+e^{mL}\right)}{\left(e^{-mL}-\frac{h}{km}e^{-mL}\right)}}$$

Let $\frac{\left(\frac{h}{km}e^{mL}+e^{mL}\right)}{\left(e^{-mL}-\frac{h}{km}e^{-mL}\right)} = z$

Therefore:

$$C_2 = \frac{\theta_b}{1+z}$$

$$C_1 = \frac{\theta_b z}{1+z}$$

Sub. C_1 & C_2 in equation (4)

$$\theta(x) = \frac{\theta_b z}{1+z} e^{-mx} + \frac{\theta_b}{1+z} e^{mx}$$

$$\frac{\theta(x)}{\theta_b} = \frac{z e^{-mx}+e^{mx}}{1+z}$$

$$\frac{\theta(x)}{\theta_b} = \frac{\frac{\left(\frac{h}{km}e^{mL}+e^{mL}\right)}{\left(e^{-mL}-\frac{h}{km}e^{-mL}\right)}e^{-mx}+e^{mx}}{1+\frac{\left(\frac{h}{km}e^{mL}+e^{mL}\right)}{\left(e^{-mL}-\frac{h}{km}e^{-mL}\right)}}$$

$$\frac{\theta(x)}{\theta_b} = \frac{\frac{\left(\frac{h}{km}e^{mL}+e^{mL}\right)e^{-mx}+e^{mx}\left(e^{-mL}-\frac{h}{km}e^{-mL}\right)}{\left(e^{-mL}-\frac{h}{km}e^{-mL}\right)}}{\frac{\left(\frac{h}{km}e^{mL}+e^{mL}\right)+\left(e^{-mL}-\frac{h}{km}e^{-mL}\right)}{\left(e^{-mL}-\frac{h}{km}e^{-mL}\right)}}$$

$$\frac{\theta(x)}{\theta_b} = \frac{\left(\frac{h}{km}e^{m(L-x)}+e^{m(L-x)}\right)+\left(e^{-m(L-x)}-\frac{h}{km}e^{-m(L-x)}\right)}{\left(\frac{h}{km}e^{mL}+e^{mL}\right)+\left(e^{-mL}-\frac{h}{km}e^{-mL}\right)}$$

$$\frac{\theta(x)}{\theta_b}=\frac{\frac{h}{km}\left(e^{m(L-x)}-e^{-m(L-x)}\right)+e^{m(L-x)}+e^{-m(L-x)}}{e^{mL}+e^{-mL}+\frac{h}{km}\left(e^{mL}-e^{-mL}\right)}*\frac{\frac{1}{2}}{\frac{1}{2}}$$

$$\frac{\theta(x)}{\theta_b}=\frac{\frac{h}{km}\frac{\left(e^{m(L-x)}-e^{-m(L-x)}\right)}{2}+\frac{e^{m(L-x)}+e^{-m(L-x)}}{2}}{\frac{e^{mL}+e^{-mL}}{2}+\frac{h}{km}\frac{\left(e^{mL}-e^{-mL}\right)}{2}}$$

The final solution:

$$\frac{\theta(x)}{\theta_b}=\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{\cosh m\,(L-x)+\frac{h}{km}\sinh m\,(L-x)}{\cosh mL+\frac{h}{km}\sinh mL}$$ ... (16) (Temperature distribution)

Heat transfer:

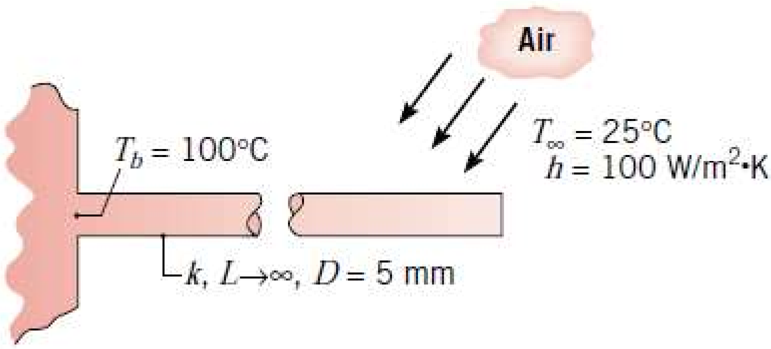
$$q_f=\sqrt{hPk\,A_c}\;\theta_b\,\frac{\sinh mL+\frac{h}{km}\cosh mL}{\cosh mL+\frac{h}{km}\sinh mL}$$ .... (17) (heat transfer)

Example:

A very long rod 5 mm in diameter has one end maintained at 100°C. The surface of the rod is exposed to ambient air at 25°C with a convection heat transfer coefficient of 100 W/m^2 .K.

Determine the heat loss from along rods constructed from pure copper, 2024 aluminum alloy, and type AISI 316 stainless steel.

Solution: go to tables in the book to get the thermal conductivity of the maintioned material of the rods.



Assumptions:

1. Steady-state conditions.

2. One-dimensional conduction along the rod.

3. Constant properties.

4. Negligible radiation exchange with surroundings.

5. Uniform heat transfer coefficient.

6. Infinitely long rod.

$$q_f = \sqrt{hPkA_c}\,\theta_b$$

Hence for copper,

$$q_f = \left[100\ \text{W/m}^2\cdot\text{K} \times \pi \times 0.005\ \text{m} \times 398\ \text{W/m}\cdot\text{K} \times \frac{\pi}{4}(0.005\ \text{m})^2\right]^{1/2} (100 - 25)^\circ\text{C}$$

$$= 8.3\ \text{W}$$ ◁

Similarly, for the aluminum alloy and stainless steel, respectively, the heat rates are $q_f = 5.6$ W and 1.6 W.

Fin Efficiency:

The temperature of a fin drops along the fin, and thus the heat transfer from the fin will be less because of the decreasing temperature difference toward the fin tip. To account for the effect of this decrease in temperature on heat transfer, we define fin efficiency as,

$$\text{Fin efficiency} = \frac{\text{actual heat transferred}}{\text{heat that would be transferred if entire fin area were at base temperature}} = \eta_f$$

Case A: Very long fin

$$q_f = kA_c m\theta_b = \sqrt{hPkA_c}\,\theta_b$$

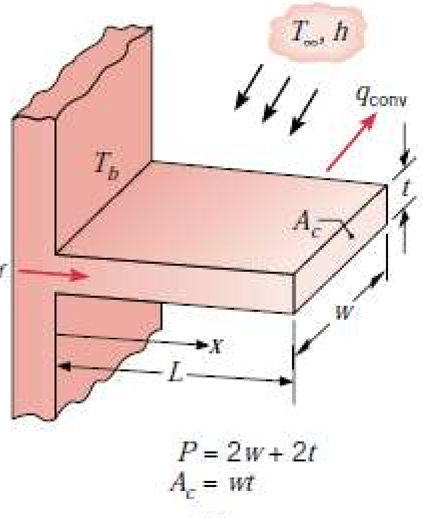
$$\eta_f = \frac{\sqrt{hPkA_c}\,\theta_b}{hPL\theta_b} = \frac{1}{L}\sqrt{\frac{kA_c}{hP}}$$

$$m^2 = \frac{hP}{kA_c}$$

Therefore:

$$\eta_f = \frac{1}{mL}$$

Case B: fin with insulated tip.



$$q_f = \sqrt{hPk\,A_c}\ \theta_b \tanh mL$$

$$\eta_f = \frac{\sqrt{hPkA_c}\theta_b \tanh mL}{hPL\theta_b}$$

$$\eta_f = \frac{\tanh mL}{mL}$$

$$mL = \sqrt{\frac{hP}{kA_c}}L = \sqrt{\frac{h(2W+2t)}{kWt}}L$$

Where:

W : is the depth of the fin, m.

t : is the thickness of the fin, m.

Now if the fin is sufficient deep, i.e the depth w large compare with t, therefore:

$$W \gg t$$

$$mL = \sqrt{\frac{2h}{kt}}L$$

$$mL = \sqrt{\frac{2h}{kt}}L * \frac{\sqrt{L}}{\sqrt{L}}$$

$$mL = \sqrt{\frac{2h}{ktL}}L^{\frac{3}{2}}$$

$$mL = \sqrt{\frac{2h}{kA_m}}L^{\frac{3}{2}}$$

Where:

$A_m = Lt$, Profile area.

Case C: finite length with heat convection from the tip of the fin.

This case can be replaced with Case B if we replaced the length by the corrected length.

Corrected fin length: $L_c = L + \frac{A_c}{p}$

$L_c = L + \frac{t}{2}$ for rectangular fin

$L_c = L + \frac{D}{4}$ for cylindrical fin

The error that results from this approximation will be less than 8 percent when

$$\left(\frac{ht}{2k}\right)^{1/2} \leq \frac{1}{2}$$

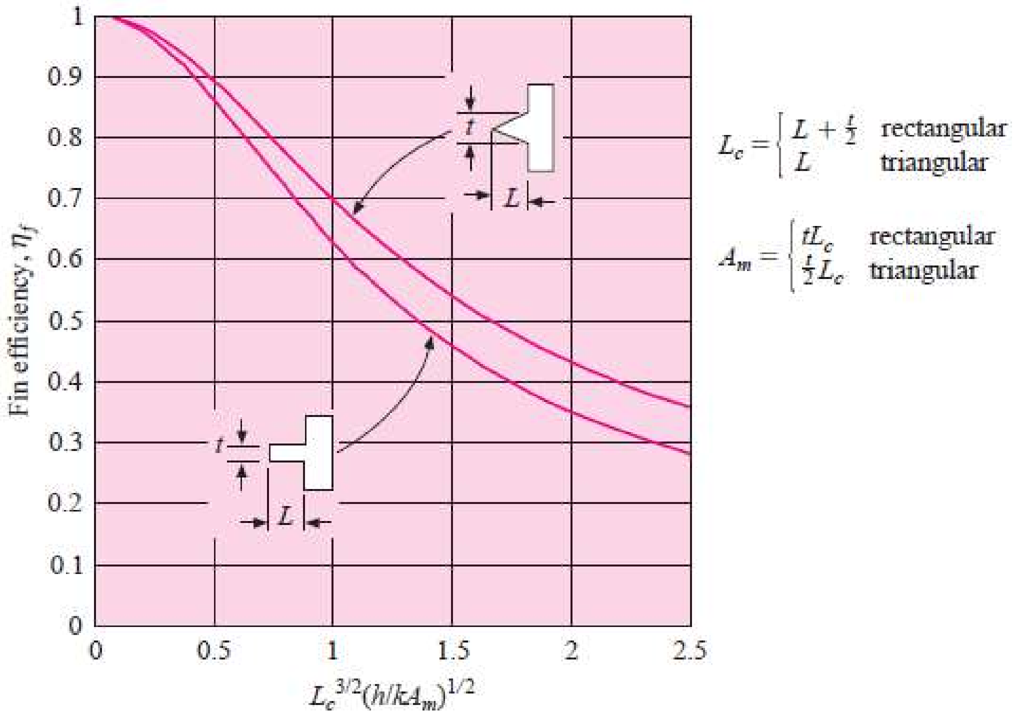


Figure 2.17: Efficiencies of straight rectangular and triangular fins.

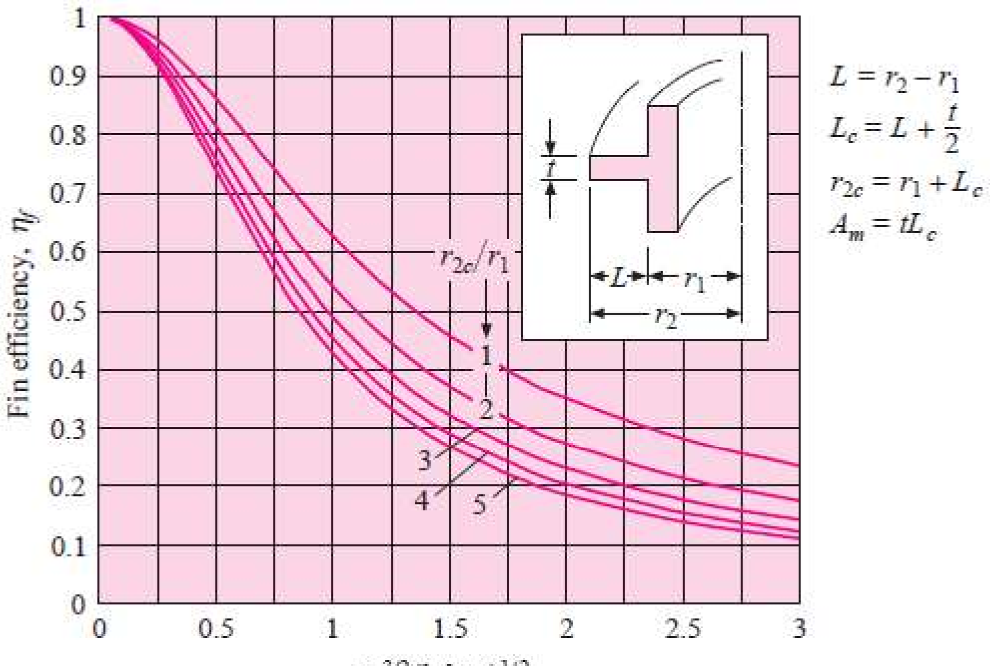
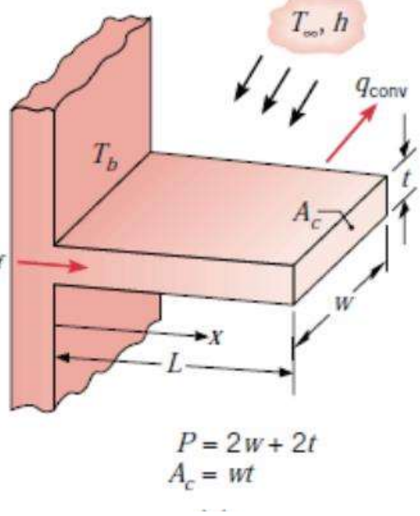


Figure 2.18: Efficiencies of circumferential fins of rectangular profile.

EXAMPLE

An aluminum fin [k =200 W/m · ◦C] 3.0 mm thick and 7.5 cm long protrudes from a wall, as in Figure below. The base is maintained at 300◦C, and the ambient temperature is 50◦C with h=10 W/m2 · ◦C. Calculate the heat loss from the fin per unit depth of material.



solution:

$L_c = L + \frac{t}{2}$ for rectangular fin

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h(2W+2t)}{kWt}}$$

$W \gg t$

$$m = \sqrt{\frac{2h}{kt}} = \left[\frac{(2)(10)}{(200)(3*10^{-3})}\right]^{1/2} = 5.774$$

The heat transfer from insulated tip fin :

$$q_f = \sqrt{hPk\,A_c}\;\theta_b \tanh mL_c$$

For a 1 m depth

$$A_c = (1)(3*10^{-3}) = 3*10^{-3}\ \text{m}^2$$

$$q_f = (5.774)(200)(\,3*10^{-3})(300-50)\tanh[(5.774)(0.0765)]$$

$$q_f = 359\ \text{W/m}$$